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TRAVELING FORCES ON STRINGS AND MEMBRANES

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Abstract—The use of explosives to simulate uniformly distributed impulsive loads is investigated theoretically for two simple structures, a stretched infinite string and a stretched infinite membrane. The action of an explosive located over a string or membrane is represented by traveling forces which are approximations to the narrow high pressure regions at the detonation fronts. The resulting velocity distributions are compared with the uniform velocity distributions caused by the same total impulse uniformly distributed over the line or area traversed by the forces. Criteria for good simulation are supplied.

INTRODUCTION

IN experimental studies [1–6] of response of structures to uniformly distributed impulsive loading, the loading is often provided by sheet explosive placed over the surface. The explosive imparts an impulse by means of a high-pressure pulse at the detonation front traveling over the surface away from the initiation point at the detonation velocity. Consequently, elements of the structure receive impulsive velocities successively and not simultaneously as desired. Good simulation of a simultaneously applied impulse over an area can be expected if the detonation velocity is sufficiently large, but some criterion is required to decide when this velocity is large enough. In this paper criteria are found for two simple structures, the stretched infinite string and the stretched infinite membrane.

The very narrow high pressure region at the detonation front is represented by a concentrated force traveling at constant velocity. On both the string and the membrane, detonation is started at one point. To the string therefore is suddenly applied a concentrated force $2P$ which separates immediately into two equal forces P which proceed to travel in opposite directions each at a constant velocity V (Fig. 1). To the membrane is applied an expanding circular ring load of magnitude P per unit length, the radius of which increases at a constant velocity V (Fig. 2). Of particular interest are the displacement and velocity distributions imparted to the string and membrane when the loads are moving supersonically relative to the wave velocities. The velocity distributions are compared with the constant velocity distribution resulting from the whole impulse being applied uniformly over the length or area traversed by the load.

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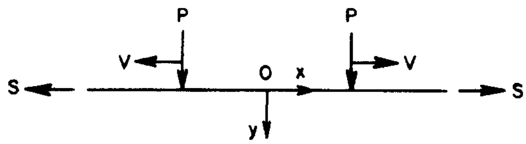

FIG. 1. String problem.

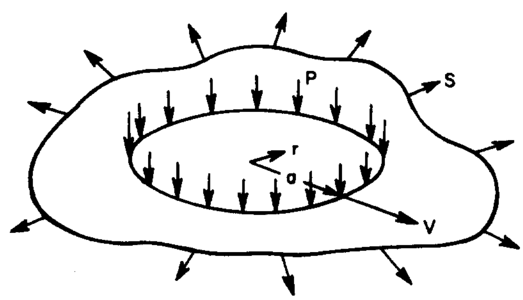

FIG. 2. Membrane problem.

Past simulation studies originated with a treatment [6] of a stretched semi-infinite string subjected to a concentrated force which runs on to it at the support and travels at a constant velocity. It was shown that the velocity distribution approaches that due to an ideal impulse covering the portion of the string traversed by the load as the ratio of the velocity of the force to the wave velocity becomes large. Later [7] the corresponding problem for the beam was studied along with the beam problem corresponding to the string problem studied here. A similar conclusion was drawn but additionally two criteria were found sufficient to produce good simulation: the detonation velocity should be supersonic and initiation should be away from a fixed support.

It is shown here that with central initiation and with practical values of wave and detonation velocities (practical values of detonation velocities are generally highly supersonic relative to string and membrane wave velocities), a uniform velocity distribution on the string and membrane is achieved. Also, results are presented to describe the distributions.

STRING THEORY

Let the two constant loads P originate at time $t = 0$ at the origin $O(x, y)$ (Fig. 1) and separate, each with a constant velocity V. Let the string have mass m per unit length, be stretched by a force s and be at rest at time $t = 0$. Then the differential equation of motion, the initial conditions, and boundary conditions are

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = -\frac{P}{mc^2}\delta(x - Vt) \qquad x > 0 \tag{1}$$

$$y(x, 0) = \frac{\partial y}{\partial t}(x, 0) = 0 \tag{2}$$

and

$$\frac{\partial y}{\partial x}(0, t) = 0 \qquad \lim_{x\to\infty} y(x, t) = 0 \tag{3}$$

where $c = (s/m)^{\frac{1}{2}}$ is the string wave velocity, and δ is the Dirac delta function.

Applying the Laplace transformation to (1) and (3), and making use of (2) gives

$$\frac{\mathrm{d}^2\bar{y}}{\mathrm{d}x^2} - (p/c)^2\bar{y} = -\frac{P}{mc^2V}\mathrm{e}^{-px/V} \tag{4}$$

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}x}(0, p) = 0 \qquad \lim_{x\to\infty} \bar{y}(x, p) = 0. \tag{5}$$

The solution of (4) satisfying (5) is

$$\bar{y}(x, p) = \frac{P}{m}\frac{1}{V^2 - c^2}(c\mathrm{e}^{-px/c} - V\mathrm{e}^{-px/V})\frac{1}{p^2}. \tag{6}$$

Inverting the transform (6), noting that there is a simple pole at the origin of the p plane, yields

$$y(x, t) = \left\{\begin{array}{ll} \frac{P}{m}\cdot\frac{(V-c)t}{V^2-c^2} & 0 < x < ct \\ \frac{P}{m}\cdot\frac{(Vt-x)}{V^2-c^2} & ct < x < Vt \\ 0 & Vt < x \end{array}\right\} 0 < c < V \tag{7}$$

$$y(x, t) = \left\{\begin{array}{ll} \frac{P}{m}\cdot\frac{(c-V)t}{c^2-V^2} & 0 < x < Vt \\ \frac{P}{m}\cdot\frac{(ct-x)}{c^2-V^2} & Vt < x < ct \\ 0 & ct < x \end{array}\right\} 0 < V < c. \tag{8}$$

Carrying out the solution for the sonic case $V = c$ leads to the result

$$y(x, t) = \left\{\begin{array}{ll} Pt/2mc & 0 < x < ct \\ 0 & ct < x \end{array}\right\} V = c$$

which predicts a growing displacement discontinuity at the force location and a uniform velocity distribution behind the force equal to $y_t = P/2mc$ ($y_t = \partial y/\partial t$). Because of the displacement discontinuity this case is not discussed further. It is sufficient to observe velocity and displacements results when V is close to c.

To explain the choice of ordinates in Figs. 3 and 4 which illustrate the results (7) and (8) consider an isolated string element of length Δx with a force P traveling over it in a time Δt. The element is supposed disconnected from the elements on either side. It receives an impulse $P\Delta t$ and, if the velocity acquired is v, the momentum acquired is $mv\Delta x = P\Delta t$. Since $\Delta t = \Delta x/V$, the velocity of the element is $v = P/mV$ and the impulse per unit length is $I = P/V$. A series of these disconnected elements resembles a string but has all the elements moving at a velocity v after the force P has traversed them, which is the same velocity distribution as that obtained by applying the impulse I per unit length to all the elements simultaneously. In the actual string this uniform velocity distribution is disturbed by the reactive impulse due to the tension and the resulting velocity distribution is the y_t obtainable from (7) or (8). Good "initial" velocity simulation is therefore achieved if the ratio y_t/v is approximately unity. Good "initial" displacement simulation requires small values of y while the force is traveling along the string. For convenience these are shown in Figs. 3(b) and 4(b) by the ratio y/vt where t is the travel time of the force but the maximum displacement is compared with the distance traversed by the force.

Figures 3(a) and 3(b) show the velocity and displacement distributions along the string according to (7) with $V = \beta c$ ($\beta > 1$). The force is moving supersonically relative to the string wave velocity. To illustrate the degree of simulation consider a practical value, $\beta = 19$. From Fig. 3(a) it is seen that the normal velocity of the string over one-nineteenth of the distance traversed by the load is 5 per cent less than that due to the same impulse uniformly distributed. Over the remaining distance it is about 0·3 per cent greater.

Good simulation also requires small displacements. Figure 3(b) shows the displacements with a maximum of $y = vt\beta/(\beta+1)$ or $y/x = (v/V)\beta/(\beta+1) = y_t/V$ where x is the load position. Hence the ratio y_t/V or v/V should be small. In order to estimate reasonable values of the ratio v/V one can equate the kinetic energy imparted to the plastic work done, assuming this to be much larger than the elastic strain-energy capacity. If the final strain is ε, the yield stress is σ_y, and the cross-sectional area is A, the energy equation is $mv^2/2 = \sigma_y \varepsilon A$. Now $m = \rho A$, and $c_y = (\sigma_y/\rho)^{\frac{1}{2}}$ is the maximum wave velocity of the string, ρ being the mass density. Thus $v = c_y\sqrt{(2\varepsilon)}$ and $v/V = \sqrt{(2\varepsilon)}/\beta_y$ where $\beta_y = V/c_y$. As a practical example consider an aluminum string stretched almost to yielding at $\sigma_y = 50{,}000\ \text{lb/in}^2$. With a mass density $\rho = 0{\cdot}00025\ \text{lb-sec}^2/\text{in}^4$ the wave velocity is about $c_y = 0{\cdot}355\ \text{mm}/\mu\text{sec}$. As an example of an explosive with one of the slower detonation velocities, oxyacetylene gas (50/50 mixture by volume) has $V = 3\ \text{mm}/\mu\text{sec}$ so that $\beta_y \approx 9$. Taking a large strain of $\varepsilon = 0{\cdot}08$ the ratio $v/V = 0{\cdot}045$. Hence the maximum displacement is approximately $y = 0{\cdot}045x$, where x is the load position, or about 5 per cent of the distance traveled by the

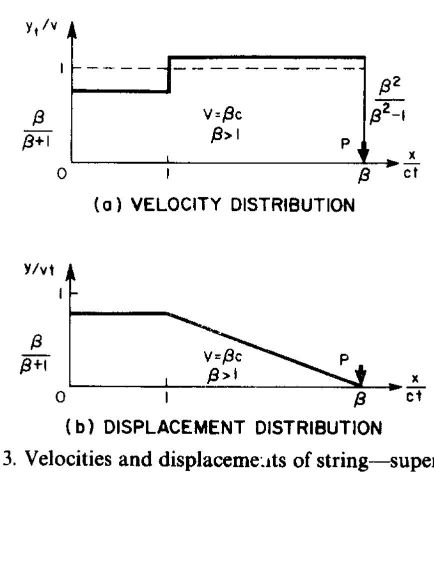

FIG. 3. Velocities and displacements of string—supersonic.

load. Using sheet explosive* with a detonation velocity of 7 mm/μsec this relation becomes $y = 0{\cdot}019x$ with $\beta_y \approx 20$.

Figures 4(a) and 4(b) show the velocity and displacement distributions along the string according to (8) with $V = \beta c\,(\beta < 1)$. For no value of $\beta < 1$ is the line $y_t/v = 1$ approximated for $0 \leq x/ct \leq \beta$ so that simulation is not possible when the load moves subsonically relative to the string wave velocity.

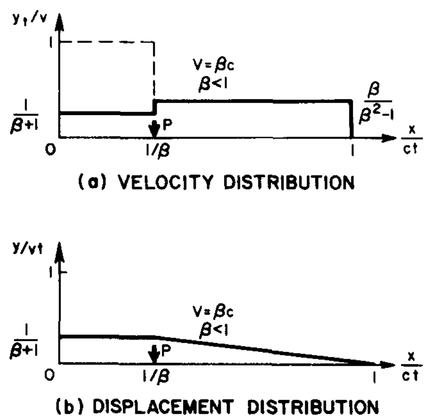

FIG. 4. Velocities and displacements of string—subsonic.

MEMBRANE THEORY

An infinite stretched membrane is subjected to a ring load of magnitude P per unit length of circumference (Fig. 2). The radius of the ring increases at a constant velocity V. Choosing the origin of the radial coordinate r at the starting or detonation point when the time is $t = 0$, the equation of motion, initial conditions and boundary conditions are

$$\frac{\partial^2 y}{\partial r^2}+\frac{1}{r}\frac{\partial y}{\partial r}-\frac{1}{c^2}\frac{\partial y^2}{\partial t^2} = -\frac{P}{mc^2}\cdot\delta(r-Vt) \tag{9}$$

$$y(r,0) = \frac{\partial y}{\partial t}(r,0) = 0 \tag{10}$$

$$\frac{\partial y}{\partial r}(0,t) = 0 \qquad \lim_{r\to\infty} y(r,t) = 0 \tag{11}$$

where y is the deflection, m the mass per unit area of membrane and $c = (S/m)^{\frac{1}{2}}$ is the membrane wave velocity. S is the tension per unit edge length.

Let $\bar{y}(\lambda, t)$ be the Hankel transform of order zero of the function $y(r, t)$. Then applying such a transformation to equations (9) and (10) gives, with the aid of (11)

$$\frac{\mathrm{d}^2\bar{y}}{\mathrm{d}t^2}+c^2\lambda^2\bar{y} = \frac{PVt}{m}J_0(\lambda Vt) \tag{12}$$

$$\bar{y}(\lambda,0) = \frac{\mathrm{d}\bar{y}}{\mathrm{d}t}(\lambda,0) = 0. \tag{13}$$

* DuPont EL-506D.

The solution of equation (12) satisfying conditions (13) is

$$\bar{y}(\lambda, t) = \frac{PV}{mc\lambda}\int_0^t \eta \sin c\lambda(t-\eta) J_0(\lambda V\eta)\,\mathrm{d}\eta \tag{14}$$

and the inverse Hankel transform of (14) is

$$y(r, t) = \frac{PV}{mc}\int_0^\infty J_0(r\lambda)\int_0^t \eta \sin c\lambda(t-\eta) J_0(\lambda V\eta)\,\mathrm{d}\eta\,\mathrm{d}\lambda. \tag{15}$$

Reversing the order of integration in (15) and setting $x = ct\lambda$ ($\mathrm{d}x = ct\,\mathrm{d}\lambda$), $\mu = \eta/t$, $\alpha = r/ct$ and $\beta = V/c$ yields

$$z(\alpha, \beta) = \int_0^1 \mu\,\mathrm{d}\mu \int_0^\infty J_0(\alpha x) J_0(\beta\mu x)\sin x(1-\mu)\,\mathrm{d}x \tag{16}$$

with

$$z = ymc^2/PVt.$$

The discontinuous integral in (16) has the following values: [8]

$$\int_0^\infty J_0(ax)J_0(bx)\sin yx\,\mathrm{d}x = \begin{cases} 0 & 0 < y < b-a \\ \dfrac{1}{2(ab)^{\frac{1}{2}}}P_{-\frac{1}{2}}(A) & b-a < y < b+a \\ \dfrac{1}{\pi(ab)^{\frac{1}{2}}}Q_{-\frac{1}{2}}(-A) & b+a < y < \infty \end{cases} \tag{17}$$

where $0 < a < b$ and

$$A = (b^2+a^2-y^2)/2ab. \tag{18}$$

In (17) $P_{-\frac{1}{2}}$ and $Q_{-\frac{1}{2}}$ are associated Legendre functions of the first and second kind and they are related to complete elliptic integrals of the first kind through the following relations. [9]

$$P_{-\frac{1}{2}}(A) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; (1-A)/2) = 2/\pi K(\sqrt{[(1-A)/2]}) \qquad -1 < A < 1$$
$$Q_{-\frac{1}{2}}(-A) = \pi/\sqrt{(-2A)}\,.\,{}_2F_1(\tfrac{1}{4}, \tfrac{3}{4}; 1; 1/A^2) = \sqrt{[2/(1-A)]}K(\sqrt{[2/(1-A)]}) \tag{19}$$
$$-\infty < A < -1$$

where ${}_2F_1$ is a hypergeometric function.

Before applying the result (17) to evaluate (18) the latter should be rewritten in the form

$$z(\alpha, \beta) = \left[\int_0^{\alpha/\beta} \mu\,\mathrm{d}\mu + \int_{\alpha/\beta}^1 \mu\,\mathrm{d}\mu\right]\int_0^\infty J_0(\alpha x)J_0(\beta\mu x)\sin x(1-\mu)\,\mathrm{d}x. \tag{20}$$

In the first double integral of (20), $0 \leq \beta\mu \leq \alpha$, while in the second, $\alpha \leq \beta\mu \leq \beta$. Making use of the relation (19), the result (17) becomes

$$\int_0^\infty J_0(\alpha x)J_0(\beta\mu x)\sin(1-\mu)x\,\mathrm{d}x = \begin{cases} 0 & \begin{cases} 0 < 1-\mu < \beta\mu-\alpha & 0<\alpha<\beta\mu \\ 0<1-\mu<\alpha-\beta\mu & 0<\beta\mu<\alpha \end{cases} \\ \dfrac{1}{\pi(\alpha\beta\mu)^{\frac{1}{2}}}K(k_1) & \begin{cases} \beta\mu-\alpha<1-\mu<\beta\mu+\alpha & 0<\alpha<\beta\mu \\ \alpha-\beta\mu<1-\mu<\alpha+\beta\mu & 0<\beta\mu<\alpha \end{cases} \\ \dfrac{1}{\pi(\alpha\beta\mu)^{\frac{1}{2}}}k_2K(k_2) & \alpha+\beta\mu<1-\mu<\infty \end{cases} \tag{21}$$

where the moduli of the complete elliptic integrals K are

$$k_1 = [(1-A)/2]^{\frac{1}{2}}$$

and

$$k_2 = [2/(1-A)]^{\frac{1}{2}}$$

in which, according to (18),

$$A = [\alpha^2+\beta^2\mu^2-(1-\mu)^2]/2\alpha\beta\mu.$$

Substituting the appropriate result from (21) in the integrals (20) leads to the following results

$$\begin{array}{ll} 1<\alpha<\beta & \pi(\alpha\beta)^{\frac{1}{2}}z = I_1\left(\dfrac{\alpha+1}{\beta+1}\right)-I_1\left(\dfrac{\alpha-1}{\beta-1}\right) \\ \begin{array}{l} 0<\alpha<1<\beta \\ 0<\alpha<\beta<1 \end{array} & \pi(\alpha\beta)^{\frac{1}{2}}z = I_1\left(\dfrac{1+\alpha}{1+\beta}\right)-I_1\left(\dfrac{1-\alpha}{1+\beta}\right)+I_2\left(\dfrac{1-\alpha}{1+\beta}\right) \\ 0<\beta<\alpha<1 & \pi(\alpha\beta)^{\frac{1}{2}}z = I_1\left(\dfrac{1-\alpha}{1-\beta}\right)-I_1\left(\dfrac{1-\alpha}{1+\beta}\right)+I_2\left(\dfrac{1-\alpha}{1+\beta}\right) \end{array} \tag{22}$$

where

$$I_1(\gamma) = \int_0^\gamma \mu K(k_1)\,\mathrm{d}\mu$$

$$I_2(\gamma) = \int_0^\gamma \mu k_2 K(k_2)\,\mathrm{d}\mu.$$

From the deflection expressions (22) the velocity distribution is readily obtained. In fact, differentiation of $y = PVtz/mc^2$ gives

$$\frac{\partial y}{\partial t} = \frac{PV}{mc^2}\left(z-\alpha\frac{\partial z}{\partial \alpha}\right) \tag{23}$$

and so expressions (22) have to be differentiated with respect to α. Singularities arise in this operation but are combined to be eliminated by considering all limiting processes in the Cauchy principal value sense. Substitution of (22) in (23) then yields the velocity distribution in the form

$$
\begin{aligned}
&1 < \alpha < \beta && \frac{mc^2}{PV}\frac{\partial y}{\partial t} = \frac{3z}{2} - \frac{1}{\pi}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\left[I_3\left(\frac{\alpha+1}{\beta+1}\right) - I_3\left(\frac{\alpha-1}{\beta-1}\right) + \frac{\pi(\alpha+1)^{\frac{1}{2}}}{2(\beta+1)^{\frac{1}{2}}} - \frac{\pi(\alpha-1)^{\frac{1}{2}}}{2(\beta-1)^{\frac{1}{2}}}\right] \\
&0 < \alpha < 1 < \beta && \\
&0 < \alpha < \beta < 1 && \frac{mc^2}{PV}\frac{\partial y}{\partial t} = \frac{3z}{2} - \frac{1}{\pi}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\left[I_3\left(\frac{1+\alpha}{1+\beta}\right) - I_3\left(\frac{1-\alpha}{1+\beta}\right) + I_4\left(\frac{1-\alpha}{1+\beta}\right) + \frac{\pi(1-\alpha)^{\frac{1}{2}}}{2(1+\beta)^{\frac{1}{2}}}\right] \\
&0 < \beta < \alpha < 1 && \frac{mc^2}{PV}\frac{\partial y}{\partial t} = \frac{3z}{2} - \frac{1}{\pi}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\left[I_3\left(\frac{1-\alpha}{1-\beta}\right) - I_3\left(\frac{1-\alpha}{1+\beta}\right) + I_4\left(\frac{1-\alpha}{1+\beta}\right) - \frac{\pi(1-\alpha)^{\frac{1}{2}}}{2(1-\beta)^{\frac{1}{2}}}\right]
\end{aligned}
\tag{24}
$$

where

$$
I_3(\gamma) = \int_0^\gamma \mu \frac{\partial}{\partial \alpha}\{K(k_1)\}\,\mathrm{d}\mu
$$

$$
I_4(\gamma) = \int_0^\gamma \mu \frac{\partial}{\partial \alpha}\{k_2 K(k_2)\}\,\mathrm{d}\mu .
$$

It is of interest to give the velocity expressions for the center of the membrane since they can be obtained explicitly. Setting $r = 0$ in (15) and continuing the derivations as before leads to

$$
\frac{mc^2}{PV}\frac{\partial y}{\partial t}(0, t) = \begin{cases} [1 - \{\pi/2 - \sin^{-1}(1/\beta)\}/(\beta^2-1)^{\frac{1}{2}}]/(\beta^2-1) & \beta > 1 \\ \frac{1}{3} & \beta = 1 \\ [\{\cosh^{-1}(1/\beta)\}/(1-\beta^2)^{\frac{1}{2}} - 1]/(1-\beta^2) & \beta < 1. \end{cases} \tag{25}
$$

The solution of the sonic case $V = c$ is outlined in the Appendix. Like the string a growing displacement discontinuity is predicted at the force radius so this case is not considered further. It is sufficient to observe velocities and displacements when V is close to c.

Figures 5(a) and 5(b) show the velocity and displacement distributions over the membrane according to (22) and (24) when the force is moving supersonically relative to the membrane wave velocity. The diagrams have been drawn for the case $V = 2c$ ($\beta = 2$) but the forms are similar for all the supersonic cases ($\beta > 1$); the higher the value of β, the flatter the velocity curve. Figure 6 shows the velocity for $\beta = 5$. As in the representation of velocity and displacement distributions for the string (Figs. 3 and 4) the velocity v used to render results dimensionless is the velocity that would be acquired by all the elements if they were disconnected from each other, that is, $v = P/mV$. Hence good simulation of impulse applied instantaneously over the circular area swept out by the detonation front is obtained if the curve y_t/v approximates the line $y_t/v = 1$ ($y_t \equiv \partial y/\partial t$). Additionally, for good simulation the displacements acquired while the load is acting should be very small.

Figures 7(a) and 7(b) show the velocity and displacement distributions over the membrane according to (22) and (24) when the force is moving subsonically relative to the membrane wave velocity. The diagrams have been drawn for the case $V = c/2$ ($\beta = 1/2$) but the forms are similar for all the subsonic cases ($\beta < 1$). Because of the significant

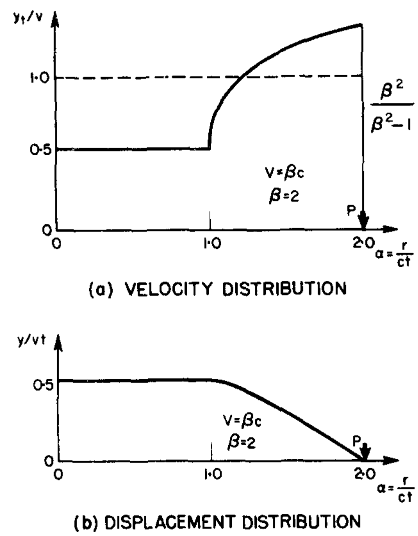

FIG. 5. Velocities and displacements of membrane—supersonic.

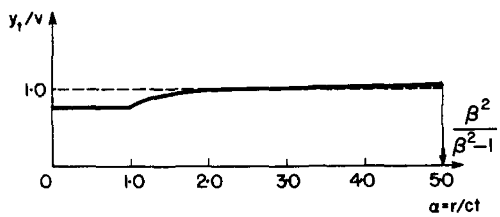

FIG. 6. Velocity distribution on membrane—supersonic $\beta = V/c = 5$.

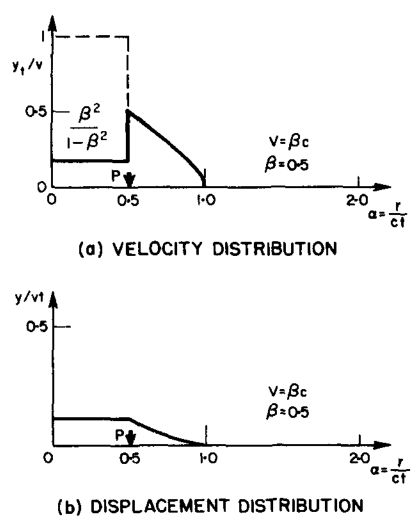

FIG. 7. Velocities and displacements of membrane—subsonic.

disturbances running ahead of the load, good simulation is never possible with subsonic loads. Figure 7(a) shows that the line $y_t/v = 1$ is not approximated.

In all supersonic cases the maximum velocity occurs under the load at $r = Vt$ and the minimum velocity is the velocity in the region $0 < r < ct$. The maximum and minimum values of the ratio y_t/v are respectively

$$\beta^2/(\beta^2-1)$$

and

$$[1-\{\pi/2-\sin^{-1}(1/\beta)\}/(\beta^2-1)^{\frac{1}{2}}]\beta^2/(\beta^2-1)$$

(The former is obtained by letting $\alpha \to \beta$ in the first of equations (24) or from the jump conditions under the force. The latter is equation (25) with $\beta > 1$.)

Table 1 lists values of y_t/v for several values of β and shows the extent of the simulation. Although the minimum values of y_t/v require high values of β before they approximate unity, the radius of the central portion of membrane moving at this minimum velocity is $1/\beta$ times the radius of the loading circle.

TABLE 1. MEMBRANE—MAXIMUM AND MINIMUM VALUES OF y_t/v

$\beta = V/c$ $\beta > 1$	$(y_t/v)_{max}$ $= \beta^2/(\beta^2-1)$ $\alpha = \beta \quad r = Vt$	$(y_t/v)_{min}$ $0 < \alpha < 1$ $0 < r < ct$
2	1·333	0·527
5	1·042	0·751
10	1·010	0·861
20	1·003	0·926
50	1·000	0·969
100	1·000	0·985
200	1·000	0·992
500	1·000	0·997
1000	1·000	0·998

As a practical example consider an aluminum membrane with a mass density $\rho = 0{\cdot}00025$ lb-sec²/in⁴ stretched to a stress of $\sigma = 50{,}000$ lb/in². The wave velocity $c = (S/m)^{\frac{1}{2}} = (\sigma/\rho)^{\frac{1}{2}}$ is then about 0·355 mm/μsec. The detonation velocities of oxyacetylene gas (50/50 mixture by volume) and sheet explosive* are approximately 3 mm/μsec and 7 mm/μsec giving values for $\beta = V/c$ of about 9 and 20 respectively. From Table 1 the minimum values of y_t/v corresponding to $\beta = 10$ and 20 are low by about 14 per cent and 7 per cent, but exist only in central circles of radii 1/9th and 1/20th of the loading circle. The less the initial stretching of a membrane (or string) the higher is the value of β, for a given explosive, and hence the better is simulation.

To estimate the deflections which may exist while the load is still acting on the structure the initial kinetic energy is equated to the final plastic work, assuming the latter much

* DuPont EL-506D.

greater than the elastic strain-energy capacity of the membrane. If each element has a final strain of ε, the yield stress is σ_y and the membrane depth is d, then the energy equation is approximately $mv^2/2 = 2\sigma_y\varepsilon d$. Now $m = \rho d$, and $c_y = (\sigma_y/\rho)^{\frac{1}{2}}$ is the maximum wave velocity in the membrane, so $v = c_y(4\varepsilon)^{\frac{1}{2}}$ and $v/V = (4\varepsilon)^{\frac{1}{2}}/\beta_y$ where $\beta_y = V/c_y$. For a strain of 4 per cent and $\beta_y = 9$ the ratio $v/V = 0{\cdot}045$ so that, approximately, the deflection is $y = 0{\cdot}045r$ where r is the radius of the loading circle. The corresponding result for $\beta_y = 20$ is $y = 0{\cdot}019r$. The larger the value of β_y the smaller the initial deflections and hence the better the simulation.

CONCLUSIONS

It can be concluded from the results of the above analyses for strings and membranes that, for good simulation of distributed impulses with explosives, the detonation velocities V must be greater than the wave velocities c. The higher the velocity ratio V/c, the better is the simulation. For both the string and membrane, values of V/c greater than 20 give very uniform "initial" velocity distributions and, provided the required final plastic strains are not too large, give small "initial" displacements. Values of V/c greater than 20 are certainly quite practical.

The actual uniformity of the initial velocity distribution can be seen in Fig. 3(a) for the string and in Table 1 (with the aid of Figs. 5(a) and 6 for the membrane). Displacements of the string and membrane acquired during loading may be estimated by the formulas $y = \sqrt{(2\varepsilon)}x/\beta$ and $y = \sqrt{(4\varepsilon)}r/\beta$ respectively where ε is the average final strain required in an experiment, x and r are the load positions, and $\beta = V/c$ is the ratio of the detonation and wave velocity.

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APPENDIX

Membrane Solution when $V = c$

When $V = c$ the solution (15) becomes

$$y(r, t) = \frac{P}{m}\int_0^\infty J_0(r\lambda)\int_0^t n \sin c\lambda(t-\eta)J_0(\lambda c\eta)\,\mathrm{d}\eta\,\mathrm{d}\lambda. \tag{A1}$$

With the substitution $\xi = \lambda c\eta$ the finite integral in (A1) becomes a tabulated integral [10a]. The result is

$$\frac{1}{\lambda^2 c^2}\int_0^{\lambda ct} \xi \sin(\lambda ct-\xi)J_0(\xi)\,\mathrm{d}\xi = \frac{t^2}{3}J_1(\lambda ct)$$

which, when substituted in (A1) gives another tabulated integral [10b]. Hence the solution is

$$y(r, t) = \frac{Pt^2}{3m}\int_0^\infty J_0(r\lambda)J_1(\lambda ct)\,\mathrm{d}\lambda = \begin{cases} Pt/3mc & 0 < r < ct \\ 0 & ct < r. \end{cases}$$

Résumé—L'emploi d'explosifs pour simuler des charges d'impulsion uniformément distribuées est théoriquement investigué pour deux structures simples, une membrane infinie étirée. L'action d'un explosif situé sur un fil ou membrane est representée par des forces à ondes progressives qui sont des approximations aux régions étroites à haute pression des fronts de détonation. Les distributions de vélocité y résultant sont comparées aux distributions de vélocité uniforme causée par le même total d'impulsion distribué uniformément sur la ligne ou région traversée par les forces. Des critères de bonne simulation sont fournis.

Zusammenfassung—Die Verwendung von Sprengstoffen zur Nachahmung gleichmässig verteilter Stosskräfte wird theoretisch für zwei einfache Strukturen untersucht. Die Wirkung des Sprengstoffes der sich über einer gespannten Linie oder Membrane befindet wird durch wandernde Kräfte dargestellt, die die engen Hochdruckbereiche der Detonationsfront annähern. Die sich ergebenden Geschwindigkeitsverteilungen werden mit den gleichmässigen Geschwindigkeitsverteilungen verglichen, die entstehen wenn dieselbe Kraft gleichmässig über die Linie oder Fläche verteilt ist. Bedingungen für die gute Simulation werden gegeben.

Абстракт—Применение взрывчатых веществ для симулирования равномерно распределённых побуждаемых импульсом нагрузок теоретически исследуется для двух простых структур растянутой бесконечной мембраны. Действие взрывчатого вещества, расположенного над шнуром или мембраной представлено перемещающимися силами, которые представляют из себя приближения к узким районам высокого давления у фронтов детонации. Получающиеся в результате распределения скорости сравниваются с распространениями равномерной скорости, причинёнными тем же полным импульсом, равномерно распределённым над линией или областью, пересекаемой силами. Даётся критерий для хорошей симуляции.